Relaxation of the quantum Brownian oscillator

1984 J. Phys. A: Math. Gen. 17 L419

(http://iopscience.iop.org/0305-4470/17/8/005)

J. Phys. A: Math. Gen. 17 (1984) L419–L422. Printed in Great Britain

LETTER TO THE EDITOR

Relaxation of the quantum Brownian oscillator

Rodolfo Figari†, Patrizia Ruggiero† and Marco Zannetti‡§

† Istituto di Fisica Teorica, Mostra d'Oltremare, pad 19, 80125 Napoli, Italy, and Istituto Nazionale di Fisica Nucleare

‡ IBM Zurich Research Laboratory, 8803 Rüschlikon, Switzerland

Received 13 March 1984

Abstract. The description of a quantum Brownian oscillator based on stochastic quantisation is applied to the non-equilibrium relaxation process. The quantum fluctuation–dissipation theorem is derived in a form which relates only the thermal component of the fluctuations to the response function. In the zero-temperature limit, it is shown that the transient is described by processes related to the solution of the nonlinear Schrödinger equation.

The linear response functions and the equilibrium fluctuations of a thermodynamic system are connected by the fluctuation–dissipation theorem (FDT). The physical meaning of the theorem is quite clear in classical statistical mechanics, where dissipation and fluctuations are both due to the interaction with the degrees of freedom of the environment. However, in the quantum case the interpretation is less straightforward, since the fluctuations cannot be ascribed just to the thermal bath, but also have an intrinsic origin which is not related in any way to dissipative effects. As a result, the correlation functions are not uniquely defined and the theorem takes different forms according to which definition is used: symmetrised, canonical (Kubo 1966) or normally ordered (Ford *et al* 1965) correlation functions are related in as many different ways to the response functions. Ultimately, the differences in the form of the theorem should be ascribed to the different treatments of the zero-point fluctuations in the definition of the correlation functions, but the physical picture remains hidden in the formalism.

Recently the quantum Brownian oscillator has been treated by means of stochastic quantisation (Ruggiero and Zannetti 1982). In the light of the previous considerations, it is then interesting to derive the FDT within the new framework. Indeed, it turns out that the stochastic treatment, although limited to the harmonic oscillator, yields the quantum FDT in a form whose physical meaning is transparent and quite instructive.

In the stochastic formulation of quantum mechanics (Nelson 1966, 1967), the pure states of an isolated system are described by Markov random processes. This approach can be extended to the thermal equilibrium states by the introduction of non-Markovian processes. For the quantum Brownian oscillator in interaction with a thermal bath (the non-dissipative case has been treated by Guerra and Loffredo (1981)) one obtains a stochastic process $x(t)$ which is the sum of two independent processes $q(t)$, $\xi(t)$

§ Permanent address: Dipartimento di Fisica, Università di Salerno, 84100 Salerno, Italy.

obeying the set of stochastic differential equations

$$dq = (p/m)\,dt, \tag{1a}$$

$$dp = -m\omega^2 q\,dt - \gamma p\,dt + \sqrt{D}\,dw, \tag{1b}$$

$$d\xi = -\omega\xi\,dt + (\hbar/m)^{1/2}\,d\eta, \tag{1c}$$

where m, ω are the mass and the frequency of the oscillator. The processes $q(t)$ and $\xi(t)$ may be regarded as the thermal and the quantum fluctuations, respectively, since (1a) and (1b) are identical to the classical Ornstein–Uhlenbeck equations, while (1c) is the Nelson equation for the ground-state process of the isolated oscillator. The phenomenological friction and diffusion coefficient γ and D satisfy the Einstein relation $D = 2m\gamma/c(\omega)$ with

$$c(\omega) = (e^{\beta\hbar\omega} - 1)/\hbar\omega \tag{2}$$

in order to have the stationary probability density

$$\rho(x) = (\hat{\beta}m\omega^2/2\pi)^{1/2}\exp(-\tfrac{1}{2}\hat{\beta}m\omega^2x^2) \tag{3}$$

with $\hat{\beta} = (2/\hbar\omega)\tanh(\beta\hbar\omega/2)$. Finally, $w(t)$, $\eta(t)$ are two independent Wiener processes with expectations

$$\langle dw\rangle = \langle d\eta\rangle = 0, \qquad \langle dw^2\rangle = \langle d\eta^2\rangle = dt.$$

We remark, for future reference, that when the interaction with the bath is switched off (γ, $D \to 0$), (1a) and (1b) become the deterministic equations of the classical oscillator and the corresponding process $x(t) = q(t) + \xi(t)$ recovers Markovianity. In fact, this is the Nelson stochastic process associated to a Glauber coherent state for the quantum undamped oscillator (Ruggiero and Zannetti 1982).

Since we have the explicit equations of motion, the derivation of the FDT is quite simple. Let us consider the relaxation process occurring when the system is initially prepared in a non-equilibrium state. This is realised by assuming that a constant force λ, acting on the system for $t < 0$, is suddenly switched off at $t = 0$. Namely, for $t < 0$, the oscillator is in the stationary equilibrium state with probability density

$$\rho_\lambda(x) = (\hat{\beta}m\omega^2/2\pi)^{1/2}\exp[-\tfrac{1}{2}\hat{\beta}m\omega^2(x - \langle x\rangle_\lambda)^2] \tag{4}$$

where $\langle x\rangle_\lambda = \lambda/m\omega^2$ is the non-vanishing expectation value of the position. At $t = 0$, when λ is set to zero, $\rho_\lambda(x)$ becomes the non-equilibrium initial density of the transient governed by (1). Eventually, for large times the asymptotic equilibrium density in (3) is reached.

We are interested in the response of $x(t)$, which is characterised by the relaxation function

$$\Phi_x(t) = \Delta\langle x(t)\rangle/\lambda \tag{5}$$

where $\Delta\langle x(t)\rangle = \langle x(t)\rangle - \langle x(t)\rangle_0$, and $\langle\cdot\rangle_0$ denotes the average in the unperturbed equilibrium state. Now, from the definition of $x(t)$ we have $\Delta\langle x(t)\rangle = \Delta\langle q(t)\rangle + \Delta\langle\xi(t)\rangle$. However, $\xi(t)$ is the *stationary* Nelson process for the ground state, hence $\Delta\langle\xi(t)\rangle = 0$, and (5) becomes

$$\Phi_x(t) = \Delta\langle q(t)\rangle/\lambda. \tag{6}$$

On the other hand, as remarked above, the thermal fluctuation $q(t)$ is an Ornstein–Uhlenbeck process, which obeys the classical FDT (apart from a quantum correction in the prefactor)

$$\Delta\langle q(t)\rangle/\lambda = c(\omega)G_{qq}(t) \tag{7}$$

with $G_{qq}(t) = \langle q(0)q(t)\rangle_0$. Hence, inserting (7) into (6) we obtain the quantum FDT in the form

$$\Phi_x(t) = c(\omega)G_{qq}(t), \tag{8}$$

namely, the relaxation is related only to the thermal component of the fluctuations. It must be remarked that the above result is not of general validity, since the splitting of the fluctuations into a thermal and a quantum component holds for linear systems. However, this is the interesting case for the point we wish to make. In fact, given that the quantum and the thermal fluctuations are statistically independent, the form (8) of the FDT is a direct manifestation of the different physical natures of quantum and thermal fluctuations, since, as is physically intuitive, the relaxation involves only the fluctuations which are actually responsible for the dissipation.

Finally, let us consider the limit of zero temperature. Since $\lim_{T\to 0} D = 0$, equations (1) become

$$\mathrm{d}q = (p/m)\,\mathrm{d}t, \tag{8a}$$

$$\mathrm{d}p = m\omega^2 q\,\mathrm{d}t - \gamma p\,\mathrm{d}t, \tag{8b}$$

$$\mathrm{d}\xi = -\omega\xi\,\mathrm{d}t + (\hbar/m)^{1/2}\,\mathrm{d}w, \tag{8c}$$

namely, the classical-like component of the motion undergoes friction without diffusion. Therefore, the process $x(t) = q(t) + \xi(t)$ is Markovian and obeys the stochastic differential equation

$$\mathrm{d}x(t) = [p(t)/m - \omega(x(t) - q(t))]\,\mathrm{d}t + (\hbar/m)^{1/2}\,\mathrm{d}w. \tag{9}$$

The corresponding quantum state, which can be regarded as the generalisation for the damped oscillator of the Glauber coherent states, has been shown by Skagerstam (1977) to be a solution of the nonlinear Schrödinger equation. Hence, at $T = 0$ the decay toward equilibrium takes place through a pure quantum state.

One should note that the Markovian behaviour obtained at $T = 0$ is a manifestation of the approximation built in the phenomenological model, since from the microscopic analysis (Ullersma 1966, Ruggiero and Zannetti 1983) the exact process at $T = 0$ is expected to be non-Markovian. In fact, the model holds in the weak coupling limit, which becomes inconsistent at $T = 0$. It is, however, interesting that the transient behaviour predicted by the model at $T = 0$ can be put into relation with the treatment of quantum dissipative systems based on the nonlinear Schrödinger equation.

One of us (MZ) wishes to thank the IBM Zurich Research Laboratory for the hospitality extended to him during the final part of this work.

References

Ford G W, Kac M and Mazur P 1965 *J. Math. Phys.* **6** 504
Guerra F and Loffredo M I 1981 *Lett. Nuovo Cimento* **30** 81

Kubo R 1966 *Rep. Prog. Phys.* **29** 255
Nelson E 1966 *Phys. Rev.* **150** 1079
—— 1967 *Dynamical Theories of Brownian Motion* (Princeton NJ: Princeton University Press)
Ruggiero P and Zannetti M 1982 *Phys. Rev. Lett.* **48** 963
—— 1983 *Phys. Rev.* A **28** 987
Skagerstam B K 1977 *J. Math. Phys.* **18** 308
Ullersma P 1966 *Physica* **32** 27